# REPORT ON THE TORSION OF THE DIFFERENTIAL MODULE OF AN ALGEBRAIC CURVE

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*Dedicated to Professor Shreeram S. Abhyankar on the occasion of his sixtieth birthday*



Abstract. There is a conjecture, that the torsionfreeness of the module of differentials in a point of an algebraic or algebroid curve should imply that the curve is non singular at that point. A report on the main results is given.

Let $k$ be a perfect field and $R$ the local ring of a closed point of an algebraic or algebroid curve over $k$. There is a conjecture that $R$ is regular if and only if the (universally finite) differential module $\Omega_{R/k}$ is torsionfree. The nontrivial part is of course to show that for a singular point the torsion submodule $\tau(\Omega_{R/k})$ of $\Omega_{R/k}$ is not zero. Although a solution for the general case is not in sight there are many special cases which have been treated successfully. In all of these the conjecture has been found to be true. It is the purpose of this paper to give a survey on some of these results with hints concerning the proofs. For simplicity let us assume for the following that $R$ is a reduced complete analytic $k$-algebra of dimension one with maximal ideal $\mathfrak{m}$ and embedding dimension $n$, which then can be represented in the form $R = k[\![X_1, \dots, X_n]\!]/I = k[\![x_1, \dots, x_n]\!]$ , where $I$ is a reduced ideal in the formal power series ring $k[\![X_1, \dots, X_n]\!]$ . We will also restrict ourselves to the case $char\, k = 0$ , although many of the results are also valid for perfect ground fields. One can distinguish several cases:

## 1. Conditions on the number of generators of $I$ .

Let $d(R) = \mu(I) - (n-1)$ denote the *deviation* of $R$ , where $\mu(I)$ denotes the minimal number of generators of $I$ . $R$ is called a *complete intersection* if $d = 0$ and an *almost complete intersection* if $d \leq 1$ . In [Be2] the cases $d \leq 1$ were solved if $R$ is a domain. This was generalized to $d \leq 3$ in the reduced case by Ulrich [Ul1], [Ul2].

Denote by $S$ the integral closure of $R$ in its full ring of quotients $K$ , and let $D : S \to SDS = \Omega_{S/k}$ and $d : R \to RdR = \Omega_{R/k}$ be the universally finite derivations of $S$ and $R$ over $k$ respectively. Since $RdR$ and $SDS$ are both of rank 1 and $SDS$ is torsionfree (even free), the kernel of the canonical homomorphism $\varphi : RdR \to SDS$ is $T := \tau(\Omega_{R/k})$, so that we have an exact sequence

$$0 \to T \to RdR \to SDS \to SDS/RDR \to 0\,.$$

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Let $x \in R$ be a normalizing parameter and $s = k[\![x]\!]$ . Then $R$ and $S$ are finite $s$–modules and the universally finite derivations $\delta : R \to R\delta R = \Omega_{R/s}$ and $\Delta : S \to S\Delta S = \Omega_{S/s}$ coincide with the universal derivations of $R$ and $S$ over $s$ respectively. $R\delta R$ and $S\Delta S$ are finitely generated torsion modules and therefore have finite lengths as $s$–modules. Further we have $R\delta R = RdR/Rdx$ and $Rdx \cap T = 0$ and therefore $SDx/RDx \cong S/R$ . Let $\alpha$ be the natural map $R\delta R \to S\Delta S$ induced by the inclusion $R \hookrightarrow S$ . One gets a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & T \arrow[d] & \mathrm{Ker}\ \alpha \arrow[d] & \\
0 \arrow[r] & Rdx \arrow[r] \arrow[d] & RdR \arrow[r] \arrow[d, "\varphi"'] & R\delta R \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & SDx \arrow[r] \arrow[d] & SDS \arrow[r] \arrow[d] & S\Delta S \arrow[r] \arrow[d] & 0 \\
 & S/R \arrow[d] & SDS/RDR \arrow[d] & \mathrm{Coker}\ \alpha \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By the snake lemma one obtains an exact sequence

$$0 \to T \to Ker\,\alpha \to S/R \to SDS/RDR \to \mathrm{Coker}\ \alpha \to 0$$

and from this for the lengths as s-modules:

$$l(T) = l(Ker\,\alpha) - l(\mathrm{Coker}\ \alpha) + l(SDS/RDR) - l(S/R)\,.$$

But $l(Ker\,\alpha) - l(\mathrm{Coker}\ \alpha) = l(R\delta R) - l(S\Delta S)$ , so that we have

**Lemma 1.**

$$l(T) = l(R\delta R) - l(S\Delta S) + l(SDS/RDR) - l(S/R)\,.$$

**1.1. The case $d(R) \leq 1$.** One can represent $R\delta R = F/U$ where $F$ is a free $R$–module and $U$ is generated by at most $rank(F) + 1$ elements. Therefore by [Be2, Satz 2] one has $l(R\delta R) = l(D(R\delta R)^{-1}/R)$ , where $D(R\delta R)$ denotes the $0^{th}$ Fitting ideal of $R\delta R$, which is by definition also the $0^{th}$ Kähler Different $\mathfrak{D}_K(R/s)$ of $R$ over $s$ . By the general theory of the differents one has the following inclusions for the Dedekind, Kähler, and Noether differents $\mathfrak{D}_D$, $\mathfrak{D}_K$, $\mathfrak{D}_N$ and the complementary modules $R^* = \mathfrak{C}(R/s)$ and $S^* = \mathfrak{C}(S/s)$ :

$$R \subseteq S \subseteq S^* \subseteq R^* \subseteq \mathfrak{D}_D(R/s)^{-1} = \mathfrak{D}_N(R/s)^{-1} \subseteq \mathfrak{D}_K(R/s)^{-1}\,.$$

Therefore we have

$$
\begin{aligned}
l(R\delta R) = l(\mathfrak{D}_K(R/s)^{-1}/R) &\geq l(R^*/R) \\
&= l(S^*/S) + l(R^*/S^*) + l(S/R) \\
&= l(S^*/S) + 2 \cdot l(S/R) \text{ by duality.}
\end{aligned}
$$

Now $S$ is a principal ideal ring and $S\Delta S$ is of projective dimension 1 , so $l(S\Delta S) = l(S/\mathfrak{D}_K(S/s)) = l(S/\mathfrak{D}_D(S/s)) = l(S^*/S)$ , and therefore $l(R\delta R) - l(S\Delta S) \geq 2 \cdot l(S/R)$ .

If $R$ is a complete intersection one even has $R^* = \mathfrak{D}_D(R/s)^{-1}$ and $\mathfrak{D}_K(R/s) = \mathfrak{D}_D(R/s)$ by [Ku2, §10 and appendix G,G1 and G2]. Therefore $l(R\delta R) - l(S\Delta S) = 2 \cdot l(S/R)$ and we obtain

**Theorem 1.** *(*[Be2, Satz 7]*): If $d \leq 1$ then*

$$l(T) \geq l(SDS/RDR) + l(S/R)$$

*with equality if $R$ is a complete intersection.*

*Consequently: If $T = 0$ then $R$ is regular.*

There is another expression for $l(T)$ in this case due to Kunz [Ku1]: Look at the natural homomorphism of $RdR$ into the $R$-module $\omega_R$ of regular differentials:

$$c_R \colon RdR \to R^* \cdot Dx = \omega_R$$

induced by the inclusion $R \hookrightarrow Quot(R)$ . Obviously Ker $c_R = T$ , but also Coker $c_R$ is interesting in this context: One has the inclusions $R^* \cdot Dx \supseteq S^* \cdot Dx \supseteq RDR \supseteq RDx$ and so

$$
\begin{aligned}
l(\text{Coker } c_R) &= l(R^* \cdot Dx/RDR) = l(R^*/R) - l(RDR/RDx) \\
&= l(R^*/S^*) l(S^*/S) + l(S/R) + l(SDS/RDR) - l(SDS/SDx) - l(SDx/RDx) \\
&= l(S/R) + l(S\Delta S) + l(S/R) + l(SDS/RDR) - l(S\Delta S) - l(S/R) \\
&= l(SDS/RDR) + l(S/R) .
\end{aligned}
$$

If $R$ is a complete intersection then it follows with theorem 1 that $l(T) = l(\text{Coker } c_R)$ . If $R$ is only an almost complete intersection, Kunz shows in [Ku1] that $l(T) - l(\text{Coker } c_R) = l(\tau(I/I^2))$ , where $\tau(\ )$ denotes the torsion submodule. For a complete intersection $\tau(I/I^2) = 0$ by [Ku1,theorem 1], so that we have

**Theorem 1'.** *If $d \leq 1$ then*

$$l(T) = l(\text{Coker } c_R) + l(\tau(I/I^2)) \ .$$

**1.2. The case $d(R) \leq 3$.** In general it is not possible to give an inequality for the length of an $R$-module using only the $0^{th}$ Fitting ideal. So the proof from 1.1. cannot be applied. (See the examples in [B2].) But Ulrich developed a more complex formula, using a chain of certain determinantal ideals, by which a lower bound for the length of an arbitrary finitely generated $R$-module can be computed. ([Ul1, Satz 2], [Ul2, Satz 1]).

We start again with Lemma 1 , but this time we choose the parameter $x$ so that $SDR = SDx$. This is possible by [Ul1, Hilfssatz 3]. By Lemma 1 we have

$$
\begin{aligned}
l(T) &= l(R\delta R) - l(S\Delta S) + l(SDS/SDR) + l(SDR/RDR) - l(S/R) \\
&= l(R\delta R) - l(S\Delta S) + l(\underbrace{SDS/SDx}_{=S\Delta S}) + l(S \cdot RDR/RDR) - l(S/R) \text{ by choice of } x .
\end{aligned}
$$

Hence

**Proposition 1.** *(*[Ul1, Satz 3]*)*

$$l(T) = l(R\delta R) - l(S/R) + l(S \cdot RDR/RDR)$$
$$\geq l(R\delta R) - l(S/R)\,.$$

**Definition 1.** ([Ul1, Def.3]) $R$ has *minimal torsion* if and only if $l(T) = l(R\delta R) - l(S/R)$.

**Proposition 2.** *(*[Ul1, Bemerkung 3]*) If $R$ has minimal torsion then there exists a non zero divisor $y \in \mathfrak{m}$ with $y \cdot \mathfrak{m} = \mathfrak{m}^2$. If $R$ is singular then $T \neq 0$.*

*Proof.* If $R$ has minimal torsion then by proposition 1 $l(S \cdot RDR/RDR) = 0$, hence $RDR = S \cdot RDR = S \cdot Dx \cong S$. $\Longrightarrow l_R(\mathfrak{m}/\mathfrak{m}^2) = n \geq \mu_R(RDR) = \mu_R(S) = l_R(S/\mathfrak{m} \cdot S)$. Now there is a non zero divisor $y \in \mathfrak{m}$ with $S \cdot \mathfrak{m} = S \cdot y$ since $S$ is a principal ideal ring ([Ul1, Hilfssatz 2]). Then $l_R(\mathfrak{m}/y \cdot \mathfrak{m}) = l_R(S/S \cdot y) = l_R(S/S \cdot \mathfrak{m}) l_R(\mathfrak{m}/\mathfrak{m}^2)$ hence $\mathfrak{m}^2 \supseteq y \cdot \mathfrak{m} \supseteq \mathfrak{m}^2$. By theorem 8 we have then $T \neq 0$, if $R$ is singular.

Now by the above mentioned length formula of Ulrich, using also the relations between the various differents and the fact that $R^*$ as in 1.1 is a canonical $R$-ideal, Ulrich shows in [Ul1, Satz 6]:

**Proposition 3.** *If $d \leq 3$ then $l(R\delta R) \geq l(S/R) + l(S^*/S) \geq l(S/R)$.*

Now we can prove the torsion conjecture in this case:

**Theorem 2.** *(*[Ul1, Satz 6], [Ul2, Satz 3]*): If $d(R) \geq 3$ and $T = 0$ then $R$ is regular.*

*Proof.* By propositions 1 and 3 we have $0 = T \geq l(R\delta R) - l(S/R) \geq 0$, and hence $R$ has minimal torsion. Then by proposition 2 $T \neq 0$ if $R$ is not regular.

*Remark 1.* From propositions 1 and 3 we also get $0 = l(T) \geq l(S^*/S) \geq 0$, and therefore $S^* = S$. So $S$ over $s$ is unramified. If $k$ is algebraically closed and $R$ a domain, it follows immediately that $S = s$ and therefore also $R = S$ without using theorem 7.

## 2. Exact differentials, maximal torsion and quasi homogeneous singularities.

**2.1. Exact Differentials.** A second class of curve singularities for which the conjecture is true consists of all $R$ such that every differential of $RDR$ is exact:

**Theorem 3.** *(*[Po3, Theorem 3]*): Assume $RDR = DR$. If $T = 0$ then $R$ is regular.*

For the proof we need the following lemma ([Gü2, Satz 1]), which can be proved by a direct computation:

**Lemma 2.** *Let $A := k[\![Z_1, ..., Z_q]\!]$ be a formal power series ring over $k$, $t \in \mathbb{N}$, $A_t := A/(Z_1, \ldots, Z_q)^t$. Then $\dim_k \Omega_{A_t/k} = (t-1) \cdot \binom{q+t-l}{t}$.*

*Proof of theorem 3.* Let $x \in \mathfrak{m} \backslash \mathfrak{m}^2$ be a non zero divisor of $R$, $\bar{R} := R/x$ and $\bar{d} : \bar{R} \to \bar{R}\bar{d}\bar{R}$ the universally finite derivation of $\bar{R}$ over $k$. Then $\operatorname{edim} \bar{R} = \operatorname{edim} R - 1 = n - 1$, so that we can represent $\bar{R} = P/\mathfrak{b}$, $P := k[\![Y_1, ..., Y_{n-1}]\!]$ a

formal power series ring with maximal ideal $\mathfrak{M} = (Y_1, ..., Y_{n-1})$ and $\mathfrak{b}$ an ideal with $\mathfrak{b} \subseteq \mathfrak{M}^2$. Denote by $\partial : P/\mathfrak{M}^2 \to \Omega_{\frac{P/\mathfrak{M}^2}{k}}$ the universally finite derivation of $P/\mathfrak{M}^2$ over $k$. We obtain the following commutative diagram with exact rows of canonical maps:

$$\begin{array}{ccccc} R & \longrightarrow & P/\mathfrak{M}^2 & \longrightarrow & 0 \\ {\scriptstyle \bar{d}}\big\downarrow & & {\scriptstyle \partial}\big\downarrow & & \\ \bar{R}\bar{d}\bar{R} & \longrightarrow & \Omega_{\frac{P/\mathfrak{M}^2}{k}} & \longrightarrow & 0 \end{array}$$

This yields a surjection $\bar{R}\bar{d}\bar{R}/Im\,\bar{d} \to \Omega_{\frac{P/\mathfrak{M}^2}{k}}/Im\,\partial \to 0$. Now by assumption we have $0 = T = \operatorname{Ker}(RdR \to RDR)$ and $RDR = DR$. Hence $RdR = dR$ and consequently $\bar{R}\bar{d}\bar{R} = RdR/Rdx = \bar{d}\bar{R} = Im\,\bar{d}$. $\Longrightarrow Im\,\partial = \Omega_{\frac{P/\mathfrak{M}^2}{k}}$. $\Longrightarrow$ $n = \dim_k P/\mathfrak{M}^2 = \dim_k \Omega_{\frac{P/\mathfrak{M}^2}{k}} + \dim_k Ker\,\partial \geq \binom{n}{2} + 1$ by Lemma 2. $\Longrightarrow n \leq 2$. But $n = 2$ is not possible, because then $R$ is a plane curve singularity and therefore $T \neq 0$ by 1.1 . Then $n = 1$ and consequently $R$ is regular.

When is the condition $RDR = DR$ satisfied?

**2.2. Maximal Torsion.** Consider the universally finite derivation $S \to SDS$ of $S$ over $k$. Since $S$ is a direct product of formal power series rings $S_i \cong k_i[\![t_i]\!]$, where $k_i$ is an algebraic extension of $k$, $D$ is the direct product of the formal derivations $D_i\colon f(t_i) \mapsto f'(t_i) \cdot D_i t_i$. It follows by formal integration that the $D_i$ are surjective and hence also $D$ is surjective. This induces a surjective $k$-linear map $\tilde{D} : S/R \to SDS/RDR$, and so $l(S/R) = \dim_k(S/R) \geq \dim_k(SDS/RDR) = l(SDS/RDR)$. Together with lemma 1 we get

$$l(T) = l(R\delta R) - l(S\Delta S) - [\underbrace{l(S/R) - l(SDS/RDR)}_{\geq 0}] \leq l(R\delta R) - l(S\Delta S)\,.$$

**Definition 2.** ([Po3]): R has *maximal torsion* if and only if

$$l(T) = l(R\delta R) - l(S\Delta S)$$

or, equivalently, if and only if

$$l(S/R) = l(SDS/RDR)\,.$$

If $R$ has maximal torsion the conjecture is true:

**Theorem 4.** *(*[Po3, Theorem 1]*)*
*If $R$ has maximal torsion then $RDR = DR$.*
*Consequently: If $T = 0$ then $R$ is regular.*
*If $k$ is algebraically closed and $R$ is a domain then also the converse is true:*
*If $RDR = DR$ then $R$ has maximal torsion.*

*Proof.* As shown above the $k$-linear map $D\colon S \to SDS$ is surjective and therefore

$$\begin{aligned} l(S/R) &= \dim_k(S/R) \geq \dim_k(S/(R + \operatorname{Ker} D)) = \dim_k(SDS/DR) \\ &\geq \dim_k(SDS/RDR) = l(SDS/RDR)\,. \end{aligned}$$

If $R$ has maximal torsion then $l(S/R) = l(SDS/RDR)$ and therefore $DR = RDR$. Assume now that $k$ is algebraically closed and $R$ is a domain. Then Ker $D = k \subseteq R$ and therefore $l(S/R) = l(SDS/DR) = l(SDS/RDR)$ by hypothesis. So $R$ has maximal torsion.

*Remark 2.* Let $k$ be algebraically closed and $R$ a domain.
a) Let $x \in \mathfrak{m}\backslash\mathfrak{m}^2$ be a superficial element of $R$. Then one can find a uniformizing parameter $t$ for $S$ such that $x = t^{m(R)}$, where $m(R)$ denotes the multiplicity of $R$. With $s\colon = k[\![x]\!]$ and $S = k[\![t]\!]$ we get $l(S\Delta S) = l(SDt/SDt^{m(R)}) = m(R) - 1$. Therefore $R$ has maximal torsion if and only if $l(T) = l(R\delta R) - m(R) + 1$. This shows that definition 2 is equivalent to Ulrich's Definition 2 in [Ul1].
b) Zariski [Za1] considers the case of an irreducible plane algebroid curve over $k$ and shows that $l(T) \leq 2 \cdot l(S/R)$. Equality holds if and only if $RDR = DR$, which by theorem 4 means that $R$ has maximal torsion. He proves that this is the case if and only if the curve can be represented by a quasi homogeneous equation of the form $Y^p - X^q = 0$ with $(p, q) = 1$.

For upper bounds of the torsion of a plane curve in terms of the characteristic pairs see Azevedo [Az, propositions 3 and 4].

### 2.3. Quasi homogeneous Singularities.

**Definition 3.** ([Sch, 9.8] ): $R$ is called *quasi homogeneous* if there exists a surjective $R$-module homomorphism $\Omega_{R/k} \to \mathfrak{m}$.
Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in I\!N_0^n$ . A polynomial $F = \sum \alpha_{i_1 \ldots i_n} \cdot X_1^{i_1} \ldots X_n^{i_n}$ with coefficients in $k$ is called quasi homogeneous of type $\gamma$ ,if there is a $d \in I\!N$ such that for all $\alpha_{i_1 \ldots i_n} \neq 0$ one has $\sum \gamma_j \cdot i_j = d$. In this case $d$ is called the degree of $F$.

From [K-R, Satz 2.1. and Satz 3.1.] one obtains:

*Remark 3.* : If $I$ is generated by polynomials, then $R$ is quasi homogeneous if and only if $I$ is generated by quasi homogeneous polynomials of a fixed type $\gamma$. For an irreducible $R$ this is equivalent to $R$ being isomorphic to the analytic semigroup ring $k[\![H]\!]$ for the value semigroup $H$ of $R$.

**Theorem 5.** *(*[Po3, theorem 2]*): If $R$ is quasi homogeneous and $I$ generated by polynomials then $R$ has maximal torsion.*

*Consequently: If $T = 0$ then $R$ is regular.*

*Proof.* : (For a different proof of the second assertion without the assumption of $I$ being generated by polynomials see Scheja [Sch], Satz 9.8.) By hypothesis there is a $R$-linear map $\psi : RdR \to \mathfrak{m}$ , which by [K-R], proof of Satz 2.1, can be chosen such that $\psi(dx_i) = \gamma_i x_i$ with $\gamma_i \in \mathbb{N}$ for $i = 1, ..., n$. Since $RdR$ and $\mathfrak{m}$ are both of rank 1 and $\mathfrak{m}$ is torsionfree we have Ker $\psi = T$. We may assume that $x = x_1$ is a superficial element of degree 1 of $R$. Since $Rdx \cap T = 0$ we have an exact sequence $0 \to T \to RdR/Rdx \to \mathfrak{m}/Rx \to 0$. Therefore $l(T) = l(RdR/Rdx) - l(\mathfrak{m}/Rx) = l(R\delta R) - m(R) + 1$. Now by remark 2 a) $R$ has maximal torsion.

**2.4. The value semigroup.** Let, as before, $R$ be a domain and $k$ algebraically closed. Then $S = k[\![t]\!]$ is a discrete valuation ring. Let $\nu$ denote the normed valuation with $\nu(t) = 1$. $H = \{\nu(y) | y \in R\backslash 0\}$ is called the value semigroup of $R$. Since $SDS = SDt$ every $\omega \in RDR$ is of the form $\omega = z \cdot Dt$ with $z \in S$, and so we can define $\nu(\omega)\colon = \nu(z) + 1$. This definition is independent of the choice of $t$. For

all $y \in \mathfrak{m}$ we then have $\nu(Dy) = \nu(y)$ and therefore $\nu(RDR) \supseteq \nu(\mathfrak{m})$, but there may be elements $\omega \in RDR$ with $\nu(\omega) \notin \nu(\mathfrak{m})$ . Yoshino calls them exceptional differentials ([Yo, Def. 2.3.]). If $RDR = DR$ then obviously $\nu(RDR) = \nu(\mathfrak{m})$. The converse follows from [Za1, proof of corollary 3]. So we obtain

**Theorem 6.** *(*[Ul1, Satz 4]*): Let $k$ be algebraically closed, and $R$ a domain. Then $\nu(RDR) = \nu(\mathfrak{m})$ if and only if $RDR = DR$.*

*Consequently: If $\nu(RDR) = \nu(\mathfrak{m})$ and $T = 0$ then $R$ is regular.*

This was also stated in theorem 4.1 of [Yo], but there the proof of proposition 3.3, which is used in the proof, is wrong.

## 3. Conditions on the embedding dimension, the index of stability, and the multiplicity

**3.1. Low embedding dimension.** In the case of $n = \operatorname{edim} R = 2$ the ring $R$ represents a plane curve singularity and therefore $R$ is a complete intersection. Then the conjecture is true by theorem 1. But also in the cases $n = 3$ and $n = 4$ one has the following results by Herzog [He2, Satz 3.2 and Satz 3.3.], that are obtained using properties of the Koszul complex for which we refer the reader to [He2].

**Theorem 7.**
*a) If $n \leq 3$ and $T = 0$ then $R$ is regular. b) If $n = 4$, $R$ is Gorenstein and $T = 0$ then $R$ is regular.*

**3.2. Low index of stability.** The main tool in this section is a reduction to the case of $\dim R = 0$, which was first used by Scheja in [Sch]. The proofs of the following results are very technical, and so we will mostly contend ourselves with references to the literature.
First we need a technical lemma which generalizes the well known formula for $\mu(RdR)$:

**Lemma 3.** *(*[Gü2, Lemma 1a)] *): Let $B = k[\![X_1, \ldots, X_n]\!]/\mathfrak{a}$ such that $\mathfrak{a} \subseteq (X_1, \ldots, X_n)^t$ for a $t \in I\!N$, $\mathfrak{n}$ the maximal ideal of $B$, and $\partial$ the universally finite derivation of $B$ over $k$.*
*Then for $r = t$ and $r = t + 1$ we have*

$$l_B(B\partial\mathfrak{n}^r/\mathfrak{n}^r \cdot B\partial B) = \mu_B(\mathfrak{n}^r).$$

This lemma together with lemma 2 is the main tool for proving

**Proposition 4.** *(*[Gü2, Satz 2]*): Let $\bar{R}$ with maximal ideal $\bar{\mathfrak{m}}$ be an analytic $k$-algebra with $\dim \bar{R} = 0$, and $\bar{d}$ the universally finite derivation of $\bar{R}$ over $k$. Assume that $\bar{R} = k[\![X_1, \ldots, X_n]\!]/\mathfrak{a}$ with $\mathfrak{a} \subseteq (X_1, \ldots X_n)^r$ for some $r \in I\!N$. Then*

$$l_{\bar{R}}(\bar{R}\bar{d}\bar{R}) \geq (r-1) \cdot \binom{n+r-1}{r} + \mu_{\bar{R}}(\bar{\mathfrak{m}}^r) + \mu_{\bar{R}}(\bar{\mathfrak{m}}^{r+1}).$$

*In particular one has always (with $n = \operatorname{edim} \bar{R}$)*

$$l_{\bar{R}}(\bar{R}\bar{d}\bar{R}) \geq \frac{1}{2} \cdot n \cdot (n+1) + \mu_{\bar{R}}(\bar{\mathfrak{m}}^2) + \mu_{\bar{R}}(\bar{\mathfrak{m}}^3).$$

The following easy lemma enables us to apply the preceding results to the torsion problem for one-dimensional analytic $k$-algebras:

**Lemma 4.** *(*[Ul1, Hilfssatz 8]*): Let $R$ be a one-dimensional Cohen-Macaulay ring, $y \in \mathfrak{m}$ a non zero divisor for $R$, $M$ a finitely generated $R$-module with rank $r$, and $T_y \colon = \{z | z \in M, y \cdot z = 0\}$. Then $l(M/y \cdot M) = r \cdot l(R/y \cdot R) + l(T_y)$.*

One can now show the following theorem, which is a generalization of [Ul2, Satz 7]:

**Theorem 8.** *(*[Gü2, Satz 4]*): Let $n \geq 3$, and let $T_y \colon = \{z | z \in T, y \cdot z = 0\}$. If there exists a non zero divisor $y \in \mathfrak{m}$ such that $\mathfrak{m}^4 \subseteq R \cdot y$, then:*

*(1) If* $\operatorname{edim}(R/R \cdot y) = n-1$, *then* $\quad l(T_y) = \frac{1}{2} \cdot (n-2) \cdot (n-1)$

*(2) If* $\operatorname{edim}(R/R \cdot y) = n$, *then* $\quad l(T_y) = \frac{1}{2} \cdot (n-2) \cdot (n+1)$.

*Consequently: If there exists a non zero divisor $y \in \mathfrak{m}$ with $\mathfrak{m}^4 \subseteq R \cdot y$ then: If $T = 0$ then $R$ is regular.*

*Remark 4.* ([Gü2]):

1) If one weakens the hypothesis of theorem 8 to $\mathfrak{m}^5 \subseteq R \cdot y$, one can still show $l(T_y) \geq \frac{1}{2} \cdot (n-2) \cdot (n-1) - r(R)$, where $r(R)$ denotes the type of $R$. So, if $R$ is Gorenstein and $n \geq 4$ one has again $T \neq 0$.

2) The condition $\mathfrak{m}^{t+1} \subseteq R \cdot y$ for some $t$ is satisfied for instance if $\mathfrak{m}^t$ is stable in the sense of [H-W1,definition 1.2.] (See [H-W1, remark 1.5]).
If $m(R) \leq \operatorname{edim} R + 1$ then even $\mathfrak{m}^3 \subseteq R \cdot y$ ([Ul1, Bemerkung 9 b)]): Take for $y$ a superficial element of degree one. Then $l_R(R/R \cdot y) = m(R)$ and so $l((\mathfrak{m}^2 + R \cdot y)/R \cdot y) = l(R/R \cdot y) - l(R/(\mathfrak{m}^2 + R \cdot y)) = m(R) - \operatorname{edim} R \leq 1$. It follows $\mathfrak{m} \cdot (\mathfrak{m}^2 + R \cdot y) \subseteq R \cdot y$ and therefore $\mathfrak{m}^3 \subseteq R \cdot y$.

This is a first example of a condition between the multiplicity and the embedding dimension of $R$, which will be generalized in theorem 9.
Obviously the condition $\mathfrak{m}^{t+1} \subseteq R \cdot y$ plays an important role. If $y$ is a superficial element of degree 1 then for all large $t \in \mathbb{N}$ we have $y \cdot \mathfrak{m}^t = \mathfrak{m}^{t+1}$.

**Definition 4.** The minimal $t \in N$ such that there is a superficial element (of degree 1 ) $y$ with $y \cdot \mathfrak{m}^t = \mathfrak{m}^{t+1}$ is called the *index of stability* of $R$ and is denoted by $t(R)$.

### 3.3. Relatively low multiplicity.

**Lemma 5.** *(*[Gü2, Lemma 3]*): Let $t(R) \geq 2$. Then for every superficial element $x$ of degree one $l((Rdx + x \cdot RdR)/x \cdot RdR) \geq 2$.*

Using lemma 4 and 5 together with proposition 4 one can now derive:

**Theorem 9.** *(*[Gü2, Satz 5' and 5]*):*
*Let $x$ be a superficial element of degree one, $\bar{R} \colon = R/R \cdot x$, and $\bar{\mathfrak{m}}$ the maximal ideal of $\bar{R}$.*
*If $m(R) \leq \frac{1}{2} \cdot n \cdot (n-1) + \mu_{\bar{R}}(\bar{\mathfrak{m}}^2) + \mu_{\bar{R}}(\bar{\mathfrak{m}}^3) + 1$ and $T = 0$ then $R$ is regular.*
*More generally: Let $X \in k[\![X_1, \ldots, X_n]\!]$ be a representative for $x$, and assume that $I \subseteq (X_1, \ldots, X_n)^r + (X)$ for an $r \in \mathbb{N}$, $r \geq 2$. Then:*
*If $m(R) \leq (r-1) \cdot \binom{n+r-2}{r} + \mu_{\bar{R}}(\bar{\mathfrak{m}}^r) + \mu_{\bar{R}}(\bar{\mathfrak{m}}^{r+1}) + 1$ and $T = 0$ then $R$ is regular.*

*Proof.* : Assume $R$ not regular ($n \geq 2$). By lemma 4 we have
$l(T_x) = l(RdR/x \cdot RdR) - l(\bar{R}) = l(\bar{R}d\bar{R}) + l(Rdx + x \cdot RdR/x \cdot RdR) - l(\bar{R})$.

Since $x$ is superficial we have $l(\bar{R}) = m(R)$. By hypothesis we can write $\bar{R} = R/R \cdot x = k[\![Z_1, \ldots, Z_{n-1}]\!]/\mathfrak{b}$ with $\mathfrak{b} \subseteq (Z_1, \cdots, Z_{n-1})^r$ and so by proposition 4 $l(\bar{R}d\bar{R}) \geq (r-1) \cdot \binom{n+r-2}{r} + l(\bar{\mathfrak{m}}^r) + l(\bar{\mathfrak{m}}^{r+1})$. Together with lemma 5 (we may assume $t(R) \geq 2$ because of theorem 7) we get

$$l(T) \geq l(T_x) \geq (r-1) \cdot \binom{n+r-2}{r} + l(\bar{\mathfrak{m}}^r) + l(\bar{\mathfrak{m}}^{r+1}) + 2 - m(R),$$

and if the hypothesis of the theorem is satisfied, the right hand side is positive.

*Remark 5.* (1) In [Is, theorem 1] Isogawa gives a simple proof in the special case of the above theorem that $m(R) \leq \frac{1}{2} \cdot n \cdot (n-1) - 1$.
(2) In [Po1, Satz 3.13] Pohl shows that theorem 9 can be applied for instance in the following situation: Let (with the notation in the proof of theorem 9)
$a\colon = min\{i|(Z_1, \ldots, Z_{n-1})^i \subseteq \mathfrak{b}\}$ and $c\colon = max\{j|(Z_1, \ldots, Z_{n-1})^j \subseteq \mathfrak{b}\}$.
If $a - c \leq 2$ or $R$ a Gorenstein singularity and $a - c \leq 3$ then the hypothesis of theorem 9 is satisfied with $r = c$. In particular the first condition is satisfied if $R$ has *maximal Hilbert function* (i.e. $\mu_R(\mathfrak{m}^i) = min\{\binom{n+i-1}{i}, m(R)\}$ for all $i \in I\!N$).

Combining theorem 9 with [He2, Satz 3.2 and Satz 3.3] one gets:

**Theorem 10.** *(*[Gü2, Satz 6]*): Let $R$ be a domain.*
*a) If $m(R) \leq 9$ and $T = 0$ then $R$ is regular.*
*b) If $m(R) \leq 13$, $R$ Gorenstein, and $T = 0$ then $R$ is regular.*

*Proof.* Let $x$ be a superficial element of degree one. Assume $\mathfrak{m}^3 \not\subseteq R \cdot x$ in view of theorem 8. Using the same notation as in proposition 4 we then have $\mu(\bar{\mathfrak{m}}^2) \geq 1$ and $\mu(\bar{\mathfrak{m}}^3) \geq 1$ . Now either $m(R) \leq \frac{1}{2} \cdot n \cdot (n-1) + 3 \leq \frac{1}{2} \cdot n \cdot (n-1) + \mu(\bar{\mathfrak{m}}^2) + \mu(\bar{\mathfrak{m}}^3) + 1$, and then $R$ is regular by theorem 8, or $m(R) > \frac{1}{2} \cdot n \cdot (n-1) + 3$.
On the other hand we have by hypothesis
a) $m(R) \leq 9$ and therefore $n \leq 3$ .[He2, Satz 3.3] gives our result.
b) $m(R) \leq 13$ and therefore $n \leq 4$ .
Now the result follows with [He2, Satz 3.2].

## 4. Conditions on the linkage class

In this section let $k$ be algebraically closed.
Recall the definition of linkage: Two perfect ideals $I$ and $J$ of the same grade in a Gorenstein local ring are said to be *linked* (or *1-linked*), if there exists an ideal $G$ generated by a regular sequence, $G \subset I$, $G \subset J$ , such that $I = G : J$ and $J = G : I$.
The analytic $k$-algebra $R$ is said to be *linked* (or *1-linked*) to the analytic $k$-algebra $R'$, if $R = k[\![X_1, \ldots, X_n]\!]/I$ and $R' = k[\![X_1, \ldots, X_n]\!]/J$, with $I$ linked to $J$ .
$R$ is said to be *in the same linkage class* as $R'$ (or *t-linked to* $R'$), if there exists a sequence $R = R_0, \ldots, R_t = R'$ such that $R_i$ is linked to $R_{i+1}$ for $i = 0, \ldots, t-1$.
One defines an invariant

$$\sigma(R)\colon = l(\mathfrak{K}^{-1}/R) - l(R/\mathfrak{K})$$

for any canonical ideal $\mathfrak{K} \subseteq R$ of $R$ . One has $\sigma(R) \leq l(S/R) =: \delta(R)$ and $\sigma(R) < \delta(R)$ if $R$ is singular ([Jä, Satz 1], [H-W2, pp.337/338]). If $R$ is Gorenstein (e.g. complete intersection) then $\sigma(R) = 0$ .

**Theorem 11.** *(*[H-W2, theorem]*): Let $R$ be 1-linked to $R'$ .Then with the notation of 1.1.:*

$$l(\text{Coker } c_R) + l(\tau(I/I^2)) - l(T) = \sigma(R').$$

For the proof see [H-W2]. Here we will look at some of the consequences:
It follows immediately that both sides of the equation are even-linkage invariants (that is invariants for $t$-linkage with $t$ even).
By theorem 11 one has

$$\begin{aligned} l(T) &= l(\text{Coker } c_R) + l(\tau(I/I^2) - \sigma(R') \\ &= l(SDS/RDR) + l(\tau(I/I^2) + l(S/R) - \sigma(R'). \end{aligned}$$

Now, if $\sigma(R') = \sigma(R)$ and $R$ is not regular then $l(S/R) - \sigma(R') > 0$ and hence $T \neq 0$.
If $R$ and therefore also $R' := R_1$ is in the linkage class of a complete intersection, it is easily seen that $R$ is also *evenly* linked to a complete intersection ([H-W2, p.336]). Since $\sigma$ is an even-linkage invariant one has $\sigma(R') = 0$ and therefore by the formula of theorem 11:

**Theorem 12.** *(*[H-W2, corollary 3], [H-W3, theorem 4.5]*): If $R$ is in the linkage class of a complete intersection then*

$$\begin{aligned} l(T) &= l(\text{Coker } c_R) + l(\tau(I/I^2)) \\ &= l(S/R) + l(SDS/RDR) + l(\tau(I/I^2)). \end{aligned}$$

*Consequently: If $T = 0$ then $R$ is regular.*

There is a second class of rings for which this method gives a result:

**Definition 5.** ([H-W2]): Let $R_0$ be a reduced complete analytic $k$-algebra. $R$ is called a *small extension* of $R_0$ if there exists a non zero divisor $x$ of the integral closure of $R_0$ such that $R = R_0[x]$ and $x^2 \in R_0$ .

Now let $R$ be a small extension of a complete intersection $R_0$ . It is easily seen that $R$ is linked to itself, and so any $R'$ in the linkage class of $R$ is evenly linked to $R$ . So $\sigma(R') = \sigma(R)$ and the formula of theorem 11 yields:

**Theorem 13.** *(*[H-W2, corollary 4]*): If $R$ is in the linkage class of a small extension of a complete intersection then*

$$\begin{aligned} l(T) &= l(\text{Coker } c_R) + l(\tau(I/I^2)) - \sigma(R) \\ &> l(SDS/RDR) + l(\tau(I/I^2)) \textit{ if } R \textit{ is not regular.} \end{aligned}$$

*Consequently: If $T = 0$ then $R$ is regular.*

Quite generally one can show that of two linked singular analytic $k$-algebras at least one of them must have non trivial torsion of the differential module:

**Theorem 14.** *(*[H-W2, corollary 5 and theorem on page 335]*):*
*Let $\tau(\Omega_{R/k}) = 0$, $R'$ singular and 1-linked to $R$. Then $\tau(\Omega_{R'/k}) \neq 0$ .*

In order to show this we first need:

**Lemma 6.** *(*[H-W2, corollary 2a]*): Let $R$ be 1-linked to a singular $R'$ and let $\delta(R) \geq \sigma(R')$ then $\tau(\Omega_{R/k}) \neq 0$ .*

*Proof.* By theorem 11 one has $l(T) = l(SDS/RDR) + l(\tau(I/I^2)) + \delta(R) - \sigma(R') \geq l(SDS/RDR) + l(\tau(I/I^2))$. It follows that $T \neq 0$ if $SDS \neq RDR$. Assume now that $SDS = RDR$, then also $S \cdot RDR = RDR$. We can choose $s$ as in 1.2 . Then by proposition 1 $l(T) = l(R\delta R) - l(S/R)$ and therefore $R$ has minimal torsion. By proposition 2 we then have $T \neq 0$ .

*Proof of theorem 14.* Let $R'$ be singular and 1-linked to $R$ . Since $\tau(\Omega_{R/k}) = T = 0$ we must have $\sigma(R') > \sigma(R)$ because of lemma 6. So $\delta(R') \geq \sigma(R') > \delta(R) \geq \sigma(R)$. Applying again lemma 6 with $R$ and $R'$ interchanged yields $\tau(\Omega_{R'/k}) \neq 0$.

## 5. Smoothability conditions

**Definition 6.** $R$ is called *smoothable*, if $R \cong P/J$, where $P$ is a normal analytic $k$-algebra and $J$ is generated by a regular sequence of $P$ .

In the terminology of deformation theory that means that $R$ can be deformed into a complete intersection.
Bassein [Ba] shows in the complex analytic case, generalizing a result of Pinkham [Pi] for Gorenstein singularities:

**Theorem 15.** *(*[Ba, theorem 2.4]*):*
*If $R$ is smoothable then $l(T) = \delta(R) + \dim(SDS/RDR)$. Consequently: If $T = 0$ then $R$ is regular.*

Buchweitz and Greuel ([B-G, 6.1]) weakened the condition "smoothable" to the condition, that the degree of singularity in the general fiber is at most $\frac{1}{2} \cdot \delta(R)$. In his Dissertation Koch proves a version for an arbitrary ground field $k$. He generalizes the above results so that in the complex analytic case he only requires that $R$ can be deformed into an almost complete intersection:

**Theorem 16.** *(*[Ko, Korollar 3 to Satz 10]*):*
*Let $P$ be excellent and an almost complete intersection in codimension 1, and $R = P/J$ with $J$ generated by a regular sequence of $S$.*
*Then: $l(T) \geq \delta(R) + l(SDS/RDR)$.*
*Consequently: If $T = 0$ then $R$ is regular.*

For more results concerning smoothability see [H-W3], where formulas involving the higher (analytic) derived functors $T_i(R/s, R)$ and $T^i(R/s, R)$ are obtained. In particular theorem 12 is proved there by showing that $R$ is smoothable.

## 6. Quadratic transforms

For simplicity let us assume in this section that $R$ is a domain and $k$ algebraically closed. Instead of looking at $T$ as the kernel of the natural map $RdR \to SDS$, one can also take an intermediate ring $A$ with $R \subsetneqq A \subsetneqq S$ instead of $S$ as long as one can be sure that there is a nontrivial kernel of the induced map of the differential modules. A good candidate for this is the first quadratic transform $R_1 := R\left[\frac{x_2}{x_1}, ..., \frac{x_n}{x_1}\right]$ , where we assume without loss of generality that $x_1$ is a superficial element of degree one in $R$, because a short direct computation shows that whenever $T \neq 0$ then there is also $\operatorname{Ker} \varphi \neq 0$ . The converse is trivial. Therefore:

**Proposition 3.** *(*[Be3, theorem 1]*): Let $R_1$ be the first quadratic transform of $R$ and $\varphi : \Omega_{R/k} \to \Omega_{R_1/k}$ the natural homomorphism induced by the inclusion $R \hookrightarrow R_1$. Then $T \neq 0$ if and only if* Ker $\varphi \neq 0$ .

The hope is that the behavior of the differential modules when going from $R$ to a suitable $A$ might be easier accessible than that when going from $R$ to $S$ . One would think that the length of the torsion should strictly decrease when going from a singular $R$ to its quadratic transform $R_1$ . Indeed Bertin and Carbonne show that this is so in the case of curve on a surface in a point of the curve which is a simple point of the surface ([B-C], theorem 2). If one wants only to show $T \neq 0$ it is enough to show Ker $\varphi \neq 0$ for some $A$ . By the same reasoning as in section 1 we obtain the analog of the formula in lemma 1:

$$l(\text{Ker } \varphi) = l(AD_A A/RD_A R) + l(R\delta R) - l(A\partial A) - l(A/R)\,,$$

(with $\partial$ the universal derivation of $A$ over $s$ and $D_A$ the universally finite derivation of $A$ over $k$) .

Unfortunately, so far there is no method known to achieve this goal for $R$ . But one can at least construct a ring $\tilde{R}$ with $R \subseteq \tilde{R} \subseteq R_1$ which has the same multiplicity and the same quadratic transforms as $R$ , for which by the above formula one obtains $l(\text{Ker } \tilde{\varphi}) \geq \frac{1}{2} \cdot m \cdot (m-1)$ , with $\tilde{\varphi}$ denoting the natural homomorphism $\Omega_{\tilde{R}/k} \to \Omega_{R_1/k}$. More precisely:

**Theorem 17.** *(*[Be3, corollary to theorem 2]*): Let $\tilde{R} = s + \mathfrak{m} \cdot R_1$ . Then $l(\tau(\Omega_{\tilde{R}/k})) \geq \frac{1}{2} \cdot m \cdot (m-1)$ . Moreover $R$ has the same multiplicity $m$ and the same quadratic transforms as $R$ .*

*Consequently: If $\tau(\Omega_{\tilde{R}/k}) = 0$ then $\tilde{R}$ (and therefore also $R$) is regular.*

*Proof.* By definition $s = k[\![x]\!]$ and $x$ is an element of minimal value in $R$ . We have $\mathfrak{m} \cdot R_1 = x \cdot R_1$ , therefore $x$ has minimal value in $\tilde{R}$ as well and hence $m(\tilde{R}) = m(x) = m(R)$. For the maximal ideal $\tilde{\mathfrak{m}}$ of $\tilde{R}$ one has $\tilde{\mathfrak{m}}^2 = x \cdot \mathfrak{m}$ , therefore $\tilde{R}$ has maximal embedding dimension $\text{edim}(\tilde{R}) = m(\tilde{R}) = m$. Theorem 8 then yields $l(\tau(\Omega_{\tilde{R}/k}) \geq \frac{1}{2}m(m-1)$ . A more detailed analysis in [Be3] shows that this inequality already holds for Ker $\tilde{\varphi}$.

*Remark 6.* The ring $R$ is the glueing of $\mathfrak{m} \cdot R_1$ over $\mathfrak{m}$ in the sense of Tamone [Ta2].

## 7. Equisingularity

We conclude this paper with some remarks on the torsion and the equisingularity class of $R$. Let us always assume that $R$ is irreducible and $k$ algebraically closed.

For plane curves there are many equivalent definitions of equisingularity. For a report on the fundamental papers of Zariski [Za2] and the definitions see [Ca], Chapter III-V.

The three main equisingularity classes are:
(ES1): $R$ and $R'$ are equisingular if they have the same saturation.
(ES2): $R$ and $R'$ are equisingular if they have the same multiplicity sequence with respect to quadratic transforms.
(ES3): $R$ and $R'$ are equisingular if they have the same value semigroup.

While for plane curves all three concepts are equivalent (and also equivalent to other definitions via characteristic exponents or characteristic pairs), already for curves in 3-space over the complex numbers there is no relation between the three definitions (see [Ca], examples 5.3.4 and remark 5.4.4.). But, nevertheless:

**Theorem 18.** *(*[Po1, Satz 2.6]*): For every singular $R$ there is always an $R'$ equisingular to $R$ with $\tau(\Omega_{R'/k}) \neq 0$, no matter which of the three definitions one takes.*

*Proof.* :
(ES1): There is a generic plane projection $R'$ of $R$ equisingular to $R$ , and $\tau(\Omega_{R'/k}) \neq 0$ because of edim$R' = 2$.

(ES2): Take for $R'$ the ring $\tilde{R}$ defined in section 6. By theorem 17 it is (ES2)-equisingular to $R$ and $\tau(\Omega_{R'/k}) \neq 0$ . ($R'$ has not only the same multiplicity sequence as $R$ but even the same quadratic transforms.)

(ES3): Take for $R' = k[\![H]\!]$ the analytic semigroup ring for the value semigroup $H$ of $R$ . By definition $R'$ is (ES3)-equisingular to $R$ and by remark 3 and theorem 5 we have $\tau(\Omega_{R'/k}) \neq 0$.

Unfortunately the length of the torsion is not an invariant of the equisingularity class of the curve. For instance Zariski shows in [Za1] that for a plane curves $l(T) = 2\delta(R)$ if and only if it can be represented by an equation $Y^p - X^q = 0$ with $(p, q) = 1$ , but of course there are many other plane curves $R'$ with just one characteristic pair$(p, q)$ (and therefore in the same equisingularity class as $R$ ) for which $l(T) < 2\delta(R') = 2\delta(R)$.

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